On alternative representations for knowledge spaces

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Abstract

In the context of knowledge structures, alternative representations have been obtained for the class of knowledge spaces, such as surmise mappings and entail relations. In this paper, various additional conditions on surmise mappings are introduced and their consequences for the corresponding spaces are investigated. In particular, the condition describing well-graded knowledge spaces is detected. These results are related to the mathematical theory of convex geometries. In addition, a direct 1–1 correspondence between surmise mappings and entail relations is described and finally an overview of the different representations, together with the corresponding special cases, is presented. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction and basic definitions

In a model for the assessment of knowledge, introduced and motivated in Doignon and Falmagne (1985), a particular domain of knowledge is formalized as a set of problems or items that can be asked in this domain, a person’s knowledge state in the domain as the subset of these items that this individual is capable of solving, and the knowledge structure of the domain as the collection of all possible knowledge states. Based on this formalization some knowledge assessment procedures have been developed (Falmagne and Doignon, 1988a,b), as well as procedures to construct valid knowledge structures for given domains, either by querying experts in the field (Koppen and Doignon, 1990; Koppen, 1993; Müller, 1989) or by analyses of extensive empirical data (Falmagne, 1989, 1993). An overview of the project is presented in Falmagne et al. (1990).

The description of a field of knowledge by its knowledge structure is in fact very concrete and this representation is used in the actual assessment routines. However, it can become unwieldy, since for domains of moderate size (say, 50 items) such a structure can easily contain thousands of states, enumeration of which is generally not
very enlightening. This problem is especially felt in the context of eliciting the necessary
information about the cognitive organization of a domain, from experts. These experts
cannot be asked to produce the domain’s knowledge structure by listing all states, so
here, in particular, an alternative representation is desired.

A couple of such alternative representations have been considered, such as surmise
relations and surmise mappings (Doignon and Falmagne, 1985), and entail relations
(Koppen and Doignon, 1990). They involve the imposition of some closure properties
on the knowledge structures. Let $X$ denote a domain of knowledge, that is, a set of
problems. Throughout this paper $X$ is supposed to be finite. Let $\mathcal{H} \subseteq 2^X$ denote a
knowledge structure on $X$. Then $\mathcal{H}$ is said to be closed under union if any union of states
is a state, and closed under intersection if any intersection of states is a state. In the finite
case these conditions are, respectively, equivalent to

$$K_1 \in \mathcal{H} \land K_2 \in \mathcal{H} \Rightarrow K_1 \cup K_2 \in \mathcal{H}$$

(1)

and

$$K_1 \in \mathcal{H} \land K_2 \in \mathcal{H} \Rightarrow K_1 \cap K_2 \in \mathcal{H}.$$  

(2)

The following two conditions, which are of a different nature, have played a role in the
assessment procedures and the probabilistic learning models developed by Falmagne
(1989, 1993). Two items are called indistinguishable in a knowledge structure $\mathcal{H}$ on $X$
whenever they are contained in exactly the same states of $\mathcal{H}$. The equivalence classes of
indistinguishable elements are called notions. A knowledge structure $\mathcal{H}$ is called
discriminating when it has no indistinguishable elements:

$$\{K \in \mathcal{H} : x \in K\} = \{K \in \mathcal{H} : y \in K\} \Rightarrow x = y,$$

(3)

for any $x, y \in X$. Technically, this can always be achieved by considering the knowledge
structure to be defined on the notions instead of the separate items. Whenever convenient
it will thus be assumed that all elements of $X$ are distinguishable.

A more interesting property of knowledge structures is that of well-gradedness
(Falmagne and Doignon, 1988b; Falmagne, 1989). A knowledge structure $\mathcal{H}$ on $X$ is
said to be well-graded if for any $K \in \mathcal{H}$,

$$\begin{cases}
K \neq \emptyset \Rightarrow K \setminus \{x\} \in \mathcal{H} & \text{for some } x \in K, \\
K \neq X \Rightarrow K \cup \{y\} \in \mathcal{H} & \text{for some } y \in X \setminus K.
\end{cases}$$

(4)

Alternatively, $\mathcal{H}$ is well-graded when it is the union of maximal chains of length $|X| + 1$. The
intuitive idea is that, considering these chains as the possible learning paths, the
transition in a well-graded structure from one state to the next is always obtained by
adding just one item. This seems to make sense pedagogically, at least with “item”
replaced by “notion”. Obviously, a knowledge structure that is not discriminating
cannot be well-graded, but in such a case the reduced structure, defined on the notions,
may or may not be well-graded, and it is in this context that well-gradedness seems to be
an interesting and reasonable additional condition on knowledge structures.

In Section 2, Doignon and Falmagne’s (1985) alternative representations of classes of
knowledge structures by surmise relations and surmise mappings are recapitulated. Section 3 then introduces a number of possible additional conditions for surmise mappings. The consequences of these conditions for the corresponding knowledge structures are given in Section 4, with special attention paid to the case of well-graded structures. Section 5 relates the obtained results to the mathematical theory of convex geometries. In Section 6 the alternative representation by entail relations (Koppen and Doignon, 1990) is outlined, after which Section 7 presents a direct 1–1 correspondence between surmise mappings and entail relations, without any appeal to underlying knowledge structures. The paper concludes, in Section 8, with a survey of the various alternative representations for knowledge structures and the corresponding special cases.

2. Surmise relations and surmise mappings

A knowledge structure on a given domain of knowledge $X$ represents restrictions on the possible knowledge states in $X$ in a very direct and concrete way, viz., by enumeration. However, other ways to describe such restrictions appear more natural. One obvious observation in this context is that the mastery of some problem may imply the mastery of some other problem. This idea is formalized as a binary relation $S$ on $X$, with $xSy$ interpreted as “from a correct answer to problem $y$ it may be surmised that problem $x$ will be answered correctly”. A knowledge structure $\mathcal{K}$ is said to be compatible with a relation $S$ if $\mathcal{K}$ allows for the above interpretation of $S$, that is, if $xSy$ implies that any knowledge state containing $y$ contains also $x$.

Considering now, for a fixed knowledge structure $\mathcal{K}$, the collection of relations with which $\mathcal{K}$ is compatible, the following facts may be observed: (i) this collection is nonempty, since any knowledge structure is compatible with the identity relation; (ii) it has a unique maximal element, since if $\mathcal{K}$ is compatible with a number of relations, it is also compatible with their union; (iii) this maximal element is a reflexive and transitive relation (i.e. a quasi order), since if $\mathcal{K}$ is compatible with a relation, it is also compatible with the reflexive transitive closure of that relation. This maximal quasi order compatible with $\mathcal{K}$ is called the surmise relation corresponding to $\mathcal{K}$.

Starting, on the other hand, with a fixed relation $S$ on $X$, some properties of the collection of knowledge structures on $X$ that are compatible with $S$ follow: (i) it is nonempty, since the structure $\{\emptyset, X\}$, is compatible with any relation; (ii) it has a unique maximal element, since if a number of knowledge structures are compatible with $S$, then so is their union; (iii) this maximal element is closed under union and intersection, since if $\mathcal{K}$ is compatible with $S$, then so is the structure consisting of all unions and intersections of states in $\mathcal{K}$ (with $\emptyset$ and $X$ being the union and intersection, respectively, of zero states). Thus, with any relation is associated the maximal compatible knowledge structure, which will be closed under union and intersection.

By showing that the above correspondences define a Galois connection between relations and knowledge structures, Monjardet (1970) reproduced a classical result of Birkhoff (1937) to the effect that a knowledge structure is characterized by its surmise relation if and only if it is closed under union and intersection.
Theorem 2.1. (Birkhoff, 1937) With \( x, y \in X \), the formula

\[
    xSy \leftrightarrow \{ K \in \mathcal{K} : x \in K \} \supseteq \{ K \in \mathcal{K} : y \in K \}
\]

defines an inclusion reversing isomorphism between quasi orders \( S \) and knowledge structures \( \mathcal{K} \) that are closed under union and intersection.

Note that (5) defines for any knowledge structure \( \mathcal{K} \) the surmise relation \( S \) and for any relation \( S \) the maximal compatible knowledge structure \( \mathcal{K} \).

Attractive as it may seem, representing a knowledge structure by its surmise relation \( S \) is in general too strict a model, since it implies that any item \( x \) has a unique set of prerequisite items, namely

\[
    Sx = \{ y \in X : ySx \}.
\]

Such a representation cannot deal with the quite common case of alternative ways to solve a problem, which constitute different possible sets of prerequisites for this problem. To remedy this situation Doignon and Falmagne (1985) defined the following generalization of the notion of a surmise relation. Call a surmise mapping\(^1\) on \( X \) any mapping \( \sigma \) from \( X \) into the power set of the power set of \( X \) satisfying the following axioms: for all \( x \in X \),

\[
    \sigma(x) \neq \emptyset, \quad (6a)
\]

\[
    C \in \sigma(x) \Rightarrow x \in C, \quad (6b)
\]

\[
    y \in C \in \sigma(x) \Rightarrow C' \subseteq C \text{ for some } C' \in \sigma(y), \quad (6c)
\]

\[
    C, C' \in \sigma(x) \& C \subseteq C' \Rightarrow C = C'. \quad (6d)
\]

Thus, a surmise mapping \( \sigma \) associates with any problem \( x \) a family \( \sigma(x) \) of subsets of \( X \), called the clauses for \( x \), which constitute the possible sets of prerequisites for \( x \). The idea is that the mastery of problem \( x \) entails the mastery in totality of at least one clause for \( x \). Axioms (6b) and (6c), respectively, generalize the reflexive and transitive properties of a surmise relation; Axiom (6d) reflects that a clause represents a minimal set of prerequisites. A surmise relation \( S \) is indeed a special case of a surmise mapping, since \( \sigma \) defined by \( \sigma(x) = \{ Sx \} \) for \( x \in X \) satisfies the axioms.

Doignon and Falmagne (1985) called a knowledge space on \( X \) any structure on \( X \) that is closed under union (i.e. satisfies (1)). By dropping the axiom of closure under intersection they obtained, for finite \( X \), the following generalization of Theorem 2.1.

Theorem 2.2. (Doignon and Falmagne, 1985) For any finite set \( X \) the collection of surmise mappings on \( X \) is in 1–1 correspondence with the collection of knowledge spaces on \( X \). In this correspondence a surmise mapping \( \sigma \) maps to the closure under

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\(^1\)We call here simply a surmise mapping what in Doignon and Falmagne (1985) was termed a space-like surmise mapping.
union of \( \bigcup_{x \in X} \sigma(x) \). Conversely, a knowledge space \( \mathcal{K} \) maps to the surmise mapping \( \sigma \) in which \( \sigma(x) \) is the collection of minimal states of \( \mathcal{K} \) containing \( x \).

\( B \subseteq \mathcal{K} \) is called a basis of \( \mathcal{K} \) if any state of \( \mathcal{K} \) is the union of elements of \( B \) and no element of \( B \) can be written as the union of other elements of \( B \) (i.e. \( B \) consists of the sup-irreducible elements of \( \mathcal{K} \)). Any finite knowledge space has a unique basis. The clauses of a surmise mapping constitute the basis of the corresponding knowledge space and a basis element is a clause for those of its members not contained in any properly included basis element. (By definition, any basis element contains at least one such member.)

Let \( \mathcal{K} \) be a knowledge structure on \( X \) with surmise mapping and surmise relation \( S \). Intuitively, it is clear that the problems that one can surmise from a correct answer to problem \( x \) are exactly the problems that are common to all sets of prerequisites of \( x \). Thus, for any \( x \in X \),

\[
S_x = \bigcap \sigma(x),
\]

which formally is an immediate consequence of (5) and Theorem 2.2. It is also clear that any two problems that are indistinguishable in \( \mathcal{K} \) are also indistinguishable in \( \sigma \). By (7), any two problems that are indistinguishable in \( \sigma \) are so in \( S \) and by (5) any two problems indistinguishable in \( S \) are so in \( \mathcal{K} \). Thus all forms of distinguishability coincide. As any knowledge structure can be made discriminating by defining it on the notions instead of the items, any surmise relation can be made antisymmetric. Whenever convenient, it will thus be assumed that the surmise relation of a knowledge structure is a partial order, instead of just a quasi order.

3. Extra conditions for a surmise mapping

Now a number of additional conditions are introduced that may be imposed on a surmise mapping. A need for this may arise in view of the interpretation of surmise mappings. A clause for a problem \( x \) is interpreted as a possible collection of antecedents or prerequisites for \( x \). This suggests that the system of clauses should not have too much cyclicity. After all, subjects are supposed to be able to move in some reasonable way through the corresponding knowledge space, from the null state to the master set \( X \).

Below, a number of possible definitions of “acyclicity” of a surmise mapping are considered and the next section investigates their impact on the corresponding knowledge spaces (via Theorem 2). In describing the conditions, the following two relations \( \bar{\mathcal{R}}_\sigma \) and \( \mathcal{R}_\sigma \) on \( X \), indexed by a surmise mapping \( \sigma \) on \( X \), are used: for \( x, y \in X \),

\[
x \bar{\mathcal{R}}_\sigma y \iff x \in \bigcap \sigma(y)
\]

and

\[
x \mathcal{R}_\sigma y \iff xx \in \bigcup \sigma(y).
\]

By (7), \( \bar{\mathcal{R}}_\sigma \) is simply the surmise relation of the knowledge space corresponding to \( \sigma \); \( x \mathcal{R}_\sigma y \) codes that \( x \) appears in some clause for \( y \).
The worst case of cyclic behavior cannot materialize in surmise mappings: if $x$ appears in every clause for $y$, the converse cannot hold without having $\sigma(x) = \sigma(y)$, thus causing $x$ and $y$ to be equivalent throughout. The following example shows that a situation almost as bad is possible. In examples, subsets of $X$ are written without separators and braces; that is, $abc$ stands for the subset $\{a, b, c\}$, and so on.

**Example 3.1.** Let $X = \{a, b, c, d\}$ and $\sigma$ be defined by:

$$
\sigma(a) = \{abc, abd\}, \sigma(b) = \{abc, bd\}, \sigma(c) = \{c\}, \sigma(d) = \{d\}.
$$

It is easily checked that the conditions for a surmise mapping are satisfied. Clearly, $b$ appears in every clause for $a$, and while it is not true that $a$ appears in every clause for $b$, it does appear in some clause for $b$ (namely, $abc$). In this situation, any clause for $b$ containing $a$ may be considered very unrealistic. One could forbid one problem from appearing in any clause for another problem when the latter appears in every clause of the former. Formally,

$$
xQ_a y \land yR_b x \Rightarrow x = y. \tag{8}
$$

In the next example this condition is satisfied.

**Example 3.2.** Let $X = \{a, b, c\}$ and $\sigma$ be defined by:

$$
\sigma(a) = \{ab, ac\}, \sigma(b) = \{ab, bc\}, \sigma(c) = \{c\}.
$$

Although (8) is satisfied, there is still a problem with the interpretation of this surmise mapping. The set $ab$ is a clause for $a$, and as such gives rise to the interpretation: it is possible to arrive at $a$ via $b$. In order to get to $b$, then, a clause for $b$ must be fulfilled that is contained in this clause for $a$. The only such clause, however, is $ab$ itself, which now must be interpreted as a way of getting to $b$ via $a$. To avoid this cyclic pattern the surmise mapping must be required to be exclusive in the sense that a clause cannot be shared by two distinct problems:

$$
x \neq y \Rightarrow \sigma(x) \cap \sigma(y) = \emptyset. \tag{9}
$$

The following surmise mapping is exclusive.

**Example 3.3.** Let $X = \{a, b, c, d\}$ and $\sigma$ be defined by:

$$
\sigma(a) = \{abc, ad\}, \sigma(b) = \{abd, bc\}, \sigma(c) = \{c\}, \sigma(d) = \{d\}.
$$

The above example, however, still shows some form of cycling. On the one hand there is a clause for $a$ containing $b$ ("$a$ can be reached via $b$") while on the other hand $a$ appears in a clause for $b$ ("$b$ can be reached via $a$"). To avoid this situation where with two distinct problems each appears in some clause for the other, $R_{\sigma}$ has to be antisymmetric:

$$
xR_{\sigma}y \land yR_{\sigma}x \Rightarrow x = y. \tag{10}
$$

By extension, a surmise mapping satisfying (10) is called antisymmetric.
Example 3.4. Let $X = \{a, b, c, d, e, f\}$ and $\sigma$ be defined by:
$$
\begin{align*}
\sigma(a) &= \{abd, ae\}, & \sigma(c) &= \{ace, cf\}, & \sigma(e) &= \{e\}, \\
\sigma(b) &= \{bcf, bd\}, & \sigma(d) &= \{d\}, & \sigma(f) &= \{f\}.
\end{align*}
$$

Now all “direct” cycles have indeed disappeared, but there are, in the above example, still similar cycles left, only with length exceeding 2. There is a clause for $a$ containing $b$, a clause for $b$ containing $c$ and, finally, a clause for $c$ containing $a$. Any objections to surmise mappings which are not antisymmetric also seem valid in this situation. That is, a clause for $b$

Finally consider the situation where the extension $\mathcal{D}_{\sigma} \subseteq \mathcal{R}_{\sigma}$ is trivial, i.e. $\mathcal{R}_{\sigma} = \mathcal{D}_{\sigma}$. From the definitions of $\mathcal{R}_{\sigma}$ and $\mathcal{D}_{\sigma}$ it is clear that this occurs if and only if each $x \in X$ has just one clause:

$\sigma(x) \subseteq C \Rightarrow C = C'$.  

Thus, $\sigma$ essentially coincides with the surmise relation $\mathcal{D}_{\sigma}$, as in the following example.

Example 3.5. Let $X = \{a, b, c, d, e, f\}$ and $\sigma$ be defined by:
$$
\begin{align*}
\sigma(a) &= \{abd, ae\} & \sigma(c) &= \{cf\} & \sigma(e) &= \{e\} \\
\sigma(b) &= \{bcf, bd\} & \sigma(d) &= \{d\} & \sigma(f) &= \{f\}
\end{align*}
$$

Proposition 3.7. Conditions (8)–(12) satisfy the following chain of implications:

$$(12) \Rightarrow (11) \Rightarrow (10) \Rightarrow (9) \Rightarrow (8)$$

and none of the reverse implications hold in general.

Proof. The only implication that is not completely obvious is $(9) \Rightarrow (8)$. To prove this, suppose $(9)$ and $x \mathcal{D}_{\sigma} y$ and $y \mathcal{R}_{\sigma} x$ for $x, y \in X$. The last condition means that $y \in C$ for some $C \in \sigma(x)$. By (6c) this implies $C' \subseteq C$ for some $C' \in \sigma(y)$. Now $x \mathcal{D}_{\sigma} y$ and again (6c) lead to $C'' \subseteq C' \subseteq C$, with $C, C'' \in \sigma(x)$ and $C' \in \sigma(y)$. By (6d), $C'' = C' = C$ and $(9)$

This notion of an acyclic surmise mapping was already introduced in Doignon and Falmagne (1985).
yields \( x = y \). The examples show that none of the implications can be reversed (and that (8) is not implied by the axioms for a surmise mapping).

4. The connection with the knowledge spaces

In this section we examine which knowledge spaces correspond to the extra Conditions (8)–(12) on surmise mappings. The situation is clear for the most restrictive Condition (12), where surmise mappings and surmise relations coincide. This means that Theorem 2.1 applies: the corresponding spaces are the ones that are closed under intersection.

To describe the effect of the other conditions define, for any knowledge space \( \mathcal{H} \), a collection of relations \( \{ \leq_k \}_{k \in \mathcal{H}} \):

\[
x \leq_k y \iff (y \in K' \in \mathcal{H} \& K' \subseteq K \Rightarrow x \in K').
\]

It is easy to check that any such \( \leq_k \) is a quasi order. In fact, this definition coincides with (5) if the universe of problems is restricted to \( K \). That is, \( \leq_k \) is the surmise relation of the subspace of \( \mathcal{H} \) induced by \( K \in \mathcal{H} \), by which is meant the collection (also closed under union) of states of \( \mathcal{H} \) that are subsets of \( K \). The following lemmas give an alternative definition in terms of surmise mappings and collect some immediate consequences.

**Lemma 4.1.** Let \( \mathcal{H}_\sigma \) be the knowledge space on \( X \) corresponding to the surmise mapping \( \sigma \) and let \( K \in \mathcal{H}_\sigma \). Then

\[
x \leq_k y \iff (C \in \sigma(y) \& C \subseteq K \Rightarrow x \in C).
\]

**Lemma 4.2.** Let \( \mathcal{H}_\sigma \) be the knowledge space on \( X \) corresponding to the surmise mapping \( \sigma \).

(i) If \( K_1, K_2 \in \mathcal{H}_\sigma \) and \( x, y \in K_1 \subseteq K_2 \), then \( x \leq_k y \) implies \( x \subseteq_k y \).

(ii) \( \mathcal{H}_\sigma = \leq_k \).

(iii) \( \mathcal{H}_\sigma = \bigcup_{K \in \mathcal{H}_\sigma} \leq_k \).

Notice that, as a consequence of (i) and (ii), \( x \neq y \) implies \( x \leq_k y \) for any \( K \in \mathcal{H}_\sigma \) containing both \( x \) and \( y \). Lemma 4.3 gives a direct translation of Conditions (8), (10) and (11) in terms of the knowledge space \( \mathcal{H}_\sigma \):

**Lemma 4.3.** Let \( \mathcal{H}_\sigma \) be the knowledge space corresponding to the surmise mapping \( \sigma \).

1. Condition (8) on \( \sigma \) is equivalent to the implication that \( x = y \) whenever both \( x \leq_k y \) and \( y \leq_k x \) for some \( K \in \mathcal{H}_\sigma \).

2. \( \sigma \) is antisymmetric (Condition (10)) iff \( x = y \) whenever there are \( K_1, K_2 \in \mathcal{H}_\sigma \) such that \( x \leq_k y \) and \( y \leq_k x \).

3. \( \sigma \) is acyclic (Condition (11)) iff \( x_1 = x_2 = \cdots = x_n \) whenever there are \( K_1, \ldots, K_n \in \mathcal{H}_\sigma \) such that \( x_i \leq_k x_{i+1} \) for \( i = 1, \ldots, n \) and \( x_{n+1} = x_1 \).
The translation of Condition (9) in terms of the relations \( \preceq_K \) is more interesting. Its proof uses the following lemma which shows that for knowledge spaces the criterion for well-gradedness simplifies.

**Lemma 4.4.** Let \( \mathcal{H} \) be a knowledge space on a finite set \( X \). Then \( \mathcal{H} \) is well-graded iff for any \( K \in \mathcal{H} \)

\[
K \neq \emptyset \Rightarrow (K \setminus \{x\} \in K \text{ for some } x \in K).
\]

**Proof.** The necessity is obvious from the definition (4) of well-gradedness for arbitrary structures. For the sufficiency it has to be shown that (13) implies that for any \( K \in \mathcal{H} \), \( K \neq X \), there is \( K' \in \mathcal{H} \) with \( K' \supseteq K \) and \( |K'| = |K| + 1 \). So, let \( K \neq X \) be a state of \( \mathcal{H} \). Since \( X \in \mathcal{H} \), it follows from (13) that there is a descending chain of states \( X = \bigcup_{i=0}^{m} K_i \supseteq K_{i+1} \supseteq \cdots \supseteq K_1 \supseteq K_0 = \emptyset \), with \( n = |X| \) and \( |K_i| = i \) for \( i = 0, \ldots, n \). Define \( K' = \bigcup_{i=0}^{m} K_i \), with \( m \) the number in \( \{1, \ldots, n\} \) such that \( K_{m-1} \subseteq K \) and \( K_m \supseteq K \). Since \( \mathcal{H} \) is a space, \( K' \in \mathcal{H} \), and, by definition, \( K' \supseteq K \) and \( |K'| = |K_{m-1} \cup K| + 1 = |K| + 1 \). \( \blacksquare \)

**Theorem 4.5.** Let \( \mathcal{H}_\sigma \) be the knowledge space on \( X \) generated by the surmise mapping \( \sigma \). Then the following conditions are equivalent:

(i) \( \sigma \) is exclusive;

(ii) for any \( K \in \mathcal{H}_\sigma \), \( \preceq_K \) is a partial order;

(iii) \( \mathcal{H}_\sigma \) is well-graded.

**Proof.** (i)⇒(ii): The only thing to prove is the antisymmetry of \( \preceq_K \). So, let \( K \in \mathcal{H}_\sigma \) and suppose \( x \preceq_K y \) and \( y \preceq_K x \). Then, obviously, \( y \in K \) and thus there is some clause \( C \) for \( y \) contained in \( K \). As in the proof of Proposition 3.7, the assumptions lead to the situation \( C^\sigma \subseteq C' \subseteq C \), with \( C, C^\sigma \in \sigma(x) \) and \( C' \in \sigma(y) \). Consequently, \( C = C' \in \sigma(x) \cap \sigma(y) \), and, \( \sigma \) being exclusive, this implies \( x = y \).

(ii)⇒(iii): Let \( K \) be a non-null state of \( \mathcal{H}_\sigma \). Then, \( K \) being finite, there is \( x \in K \) that is maximal in the partial order \( \preceq_K \), i.e., \( x \preceq_K y \) implies \( x = y \) for all \( y \in K \). By the definition of \( \preceq_K \), this means that for any \( y \in K \), \( y \neq x \), there is a state \( K_y \) included in \( K \) and containing \( y \), but not \( x \). But then, since \( \mathcal{H}_\sigma \) is a space, \( \bigcup_{y \neq x} K_y = K \setminus \{x\} \) is a state of \( \mathcal{H}_\sigma \) and, by Lemma 4.4, \( \mathcal{H}_\sigma \) is well-graded.

(iii)⇒(i): Suppose \( C \in \sigma(x) \cap \sigma(y) \) for some \( x, y \in X \). Then \( C \in \mathcal{H}_\sigma \) and, \( \mathcal{H}_\sigma \) being well-graded, there is \( z \in C \) such that \( C \setminus \{z\} \in \mathcal{H}_\sigma \). By the definition of \( C \), \( \{x, y\} \subseteq C \) and \( \{x, y\} \cap (C \setminus \{z\}) = \emptyset \). Consequently, \( z = x = y \) and \( \sigma \) is exclusive. \( \blacksquare \)

Thus, exclusive surmise mappings characterize the collection of well-graded knowledge spaces. Alternatively, a well-graded knowledge space is characterized by the fact that the surmise relations of all its induced subspaces are partial orders. In Section 2 it was noted that \( \bigcup_{x \in X} \sigma(x) \) covers the basis: any set in the basis of the space is a clause for some element \( x \). A surmise mapping is exclusive if and only if this covering is in fact a partitioning of the basis, induced by a kind of inverse mapping from the basis onto \( X \), assigning any basis element to the unique element of \( X \) for which it is a clause. By the
above theorem, the basis of a well-graded knowledge space is partitioned into subcollections of sets that are minimal for the various \( x \in X \).

Notice that the restriction to exclusive surmise mappings can very easily be built into the definition of surmise mappings, namely by replacing (6c) with

\[
y \in C \in \sigma(x) \Rightarrow C' \subseteq C' \subseteq C \setminus \{x\} \text{ for some } C' \in \sigma(y).
\]

5. The theory of convex geometries

The results of the preceding section appear to be related to the mathematical theory of convex geometries. Convex geometries appear in an abstract, combinatorial approach to the notion of convexity; they were introduced independently by Paul Edelman and Robert Jamison. Equivalent structures have been described by other authors under the names of “shelling structures” and “selectors”. The following brief sketch of some basic concepts of this theory is based on a joint paper by Edelman and Jamison (1985), where further references can be found.

Edelman and Jamison (1985) consider a finite set \( X \) and alignments of \( X \), that is, families of subsets of \( X \) that are closed under intersection. The subsets of \( X \) in such an alignment are called convex sets. A convex geometry on \( X \), then, is an alignment on \( X \) such that for every convex set \( C \neq X \) there is an element \( x \) not contained in \( C \) for which \( C \cup \{x\} \) is convex. (The synonym antimatroid is also used in the literature.) A copoint attached at \( x \) is a maximal convex set not containing \( x \). For every convex set \( C \) a \( C \)-factor relation is defined between elements that are not in \( C \): a pair \((x, y)\) of such elements is in this relation if and only if \( x \) is contained in any convex set containing \( C \cup \{y\} \).

There is an obvious connection between this theory and the knowledge space theory via the complementation mapping \( C \rightarrow X \setminus C \), turning any family of subsets closed under intersection into a family closed under union and vice versa. Thus, the above notions translate directly into the terminology of this paper. An alignment corresponds to a knowledge space, a convex set to a knowledge state and a convex geometry to a well-graded knowledge space. A copoint at \( x \) refers in the same way to a minimal state containing \( x \), or, in the language of surmise mappings, to a clause for \( x \). The \( C \)-factor relation on \( X \setminus C \), finally, is exactly the relation \( \preceq_x \) defined in the previous section, with \( K = X \setminus C \).

According to this translation, Theorem 4.5 is an independent rediscovery and combination of Edelman and Jamison’s (1985) Theorems 2.3 and 2.4 which consider conditions under which an alignment is a convex geometry. Their Theorem 2.3 states that this is the case if and only if all \( C \)-factor relations are partial orders; and their Theorem 2.4 states that an equivalent condition is that every copoint is attached at a unique point (i.e. the surmise mapping is exclusive).

\(^3\)This connection was pointed out to us by Bernard Monjardet and Vincent Duquenne.
6. Entail relations

Aiming to obtain an alternative generalization of surmise relations characterizing the class of knowledge spaces and suitable for the purpose of eliciting the necessary information from experts in a field, Koppen and Doignon (1990) considered relations $\mathcal{P}$ on the power set of $X$, with $A \mathcal{P} B$ interpreted as “a student unable to solve any of the items in $A$ will be unable to solve any of the items in $B$”. Given this interpretation, it is obvious that $\mathcal{P}$ should fulfill the following conditions. For all $A, B, C \subseteq X$,

$$A \supseteq B \Rightarrow A \mathcal{P} B,$$  (14a)

$$A \mathcal{P} B \& B \mathcal{P} C \Rightarrow A \mathcal{P} C,$$  (14b)

$$A \mathcal{P} B \& A \mathcal{P} C \Rightarrow A \mathcal{P} (B \cup C).$$  (14c)

Koppen and Doignon (1990) proved that these three conditions are in fact all that is needed to represent knowledge spaces. Any relation on $2^X$ satisfying Axioms (14) is called an entail relation for $X$.

**Theorem 6.1.** (Koppen and Doignon, 1990) With $A, B \subseteq X$, the formula

$$A \mathcal{P} B \iff \{K \in \mathcal{K}: K \cap A = \emptyset\} \subseteq \{K \in \mathcal{K}: K \cap B = \emptyset\}$$  (15)

defines an inclusion reversing isomorphism between the collection of entail relations $\mathcal{P}$ for $X$ and the collection of knowledge spaces $\mathcal{K}$ on $X$.

(Like Theorem 2.1 and unlike 2.2, this theorem is also valid for infinite $X$.) Note that, since (14a) implies reflexivity and (14b) is transitivity, entail relations for $X$ are quasi orders on $2^X$. Note further that the range of entail relations may be restricted to singleton sets, since for any $\mathcal{P}$ satisfying (14) it is easily derived that

$$APB \Leftrightarrow AP[y] \text{ for any } y \in B.$$  (16)

As a practical consequence, in querying experts about the entail relation for a domain, only questions of the form “when a student cannot solve any of the items in $A$, will he be unable to solve item $y$?” need be asked, rather than more elaborate ones involving two subsets $A$ and $B$. (Considerably more efficiency can be obtained, as described in Koppen, 1993, and put to practice in an application of this method by Kambouri et al., 1994.)

Analogously to (7) and (12) the surmise relation $S$ of a knowledge structure can be expressed in terms of the entail relation $\mathcal{P}$. It is just the converse of the entail relation, restricted to singleton subsets:

$$xSy \Leftrightarrow \{y\} \mathcal{P} \{x\},$$  (17)

for $x, y \in X$. This is clear from the interpretation of an entail relation, while formally it is an immediate consequence of (15), which is valid for any knowledge structure $\mathcal{K}$, not
just a space. Koppen and Doignon (1990, p. 321) showed that an entail relation \( \mathcal{P} \)
coincides with the surmise relation of the same knowledge structure if and only if
\[
A \mathcal{P} B \iff \text{for any } y \in B \text{ there is } x \in A \text{ such that } \{x\} \mathcal{P} \{y\},
\]
for all \( A, B \subseteq X \), which by (16) is equivalent to
\[
A \mathcal{P} \{y\} \iff \{x\} \mathcal{P} \{y\} \text{ for some } x \in A,
\]
for \( A \subseteq X \), \( y \in X \). Thus, entail relations \( \mathcal{P} \) satisfying (18) or (18') are the ones that correspond 1-1 with knowledge structures closed under union and intersection.

7. A direct correspondence between surmise mappings and entail relations

Both surmise mappings and entail relations were developed as alternative characterizations of knowledge spaces. This clearly implies a 1-1 correspondence between surmise mappings and entail relations, but the specification of this correspondence always involved the knowledge space as intermediary concept. In this section the equivalence of these two concepts is made almost immediate by translating them both into yet another domain, that of propositional logic. These translations will again be obtained via the corresponding knowledge space, but they then exhibit the equivalence of surmise mapping and entail relation on a syntactical level, without recourse to the knowledge space interpretation.

Let the domain be \( X = \{x_1, x_2, \ldots, x_n\} \) and define, for any \( x_i \in X \), a logical variable \( \tilde{x}_i \), that is, a variable that can take one of the two values TRUE or FALSE. The rule
\[
v_x(\tilde{x}_i) = \text{TRUE } \iff \text{ } x_i \in A.
\]
defines a 1-1 correspondence between subsets of \( X \) and valuations or truth assignments on \( \{\tilde{x}_1, \ldots, \tilde{x}_n\} \), mapping a knowledge structure \( \mathcal{K} \) on \( X \) to the collection of valuations \( \{v_K: K \in \mathcal{K}\} \). Clearly, a valuation on \( \{\tilde{x}_1, \ldots, \tilde{x}_n\} \) determines the truth value of any (syntactically well formed) logical formula in these variables.

Surmise mappings and entail relations were devised to describe valid inferences in the corresponding knowledge structure. In both cases, these are of the form \( \tilde{x} \rightarrow \varphi \), where the subformula \( \varphi \) represents the inferences to be drawn from the presence of \( x \) in a state of \( \mathcal{K} \). The very idea of a surmise mapping \( \sigma \) corresponding to \( \mathcal{K} \) was to collect in \( \sigma(x) \) all possible prerequisites of \( x \). Concretely, \( \sigma(x) = \{C_1, C_2, \ldots, C_m\} \) represents the fact that, for any \( K \in \mathcal{K} \),
\[
x \in K \Rightarrow C_1 \subseteq K \text{ or } C_2 \subseteq K \text{ or } \ldots \text{ or } C_m \subseteq K.
\]
Let the clause \( C_1 \) consist of \( k \) elements denoted by \( x_1, \ldots, x_k \) and define the formula \( \tilde{\gamma}_1 \) by
\[
\tilde{\gamma}_1 = \tilde{x}_1 \wedge \tilde{x}_2 \wedge \cdots \wedge \tilde{x}_k.
\]
For the other clauses construct similar formulae \( \tilde{\gamma}_2, \ldots, \tilde{\gamma}_m \). Then it is clear that the statement \( "C_1 \subseteq K" \) amounts to \( v_{K}(\tilde{\gamma}_1) = \text{TRUE} \). Defining
\[
\tilde{\sigma}_x = \tilde{\gamma}_1 \lor \tilde{\gamma}_2 \lor \cdots \lor \tilde{\gamma}_m,
\]
for \( x \in X \), it follows that
\[
\tilde{\sigma}_x = v_{K}(\tilde{\gamma}_1) \lor v_{K}(\tilde{\gamma}_2) \lor \cdots \lor v_{K}(\tilde{\gamma}_m).
\]

\[
\text{(21)}
\]
(19) is tantamount to requiring that, for any $K \in \mathcal{K}$,
\[ v_K(\tilde{x} \rightarrow \tilde{\sigma}_j) = \text{TRUE}. \]

Thus a surmise mapping $\sigma$ is identified with the collection of formulae $\{\tilde{\sigma}_i\}_{i \in X}$ defined by (21) and (20). In the case that $\mathcal{K}$ is a space, $\sigma$ determines $\mathcal{K}$ completely; intuitively this means that for any $x \in X$ the formula $\tilde{\sigma}_j$ represents all that can be inferred from the presence of $x$ in a state of $\mathcal{K}$. Formally this is expressed by the statement that any (other) formula $\psi$ representing a valid inference from the presence of $x$ must be logically implied by $\tilde{\sigma}_j$:
\[ (v_K(\tilde{x} \rightarrow \psi) = \text{TRUE for all } K \in K) \Rightarrow \tilde{\sigma}_j \rightarrow \psi = \text{TRUE}. \]

One can proceed similarly with the entail relation $\mathcal{P}$ corresponding to a knowledge structure $\mathcal{K}$. The interpretation of $\mathcal{A}\mathcal{P}\{x\}$ for $A \subseteq X$ and $x \in X$ is, according to (15), that no state of $\mathcal{K}$ containing $x$ is disjoint from $A$. Thus if $\{A_1, A_2, \ldots, A_j\}$ is for some $x \in X$ the collection of all subsets $A$ such that $\mathcal{A}\mathcal{P}\{x\}$, then
\[ x \in K \Rightarrow A_1 \cap K \neq \emptyset \text{ and } A_2 \cap K \neq \emptyset \text{ and } \ldots \text{ and } A_j \cap K \neq \emptyset. \]

It is no restriction to assume that the set $\{A_1, A_2, \ldots, A_j\}$ collects only the minimal subsets $A$ such that $\mathcal{A}\mathcal{P}\{x\}$, since, obviously, the truth value of the right hand side of (23) depends only on the minimal $A_j$ appearing there. Furthermore, these minimal subsets determine $\mathcal{P}$ completely, since, by (14a) and (14b), if $\mathcal{A}\mathcal{P}\{x\}$, then also $\mathcal{A}\mathcal{P}^x$ for any $A \supseteq A$. If $A_j = \{x_1, \ldots, x_b\}$, the formula, $\tilde{\delta}_j$ is defined by
\[ \tilde{\delta}_j = \tilde{x}_1 \lor \tilde{x}_2 \lor \cdots \lor \tilde{x}_b, \]
and there are similar formulae $\tilde{\delta}_2, \ldots, \tilde{\delta}_j$ for the other sets $A_j$. In this way “$A_j \cap K \neq \emptyset$” is equivalent to $v_K (\tilde{\delta}_j) = \text{TRUE}$. Defining
\[ \tilde{\rho}_j = \tilde{\delta}_1 \land \tilde{\delta}_2 \land \cdots \land \tilde{\delta}_j, \]
(23) can now be reformulated as
\[ v_K(\tilde{x} \rightarrow \tilde{\rho}_j) = \text{TRUE}, \]
for any $K \in \mathcal{K}$. Again, if $\mathcal{K}$ is a space, $\mathcal{P}$ determines $\mathcal{K}$ completely and, for any $x \in X$, $\tilde{\rho}_j$ describes all inferences from the presence of $x$ in a state of $\mathcal{K}$. The formal consequence of this is again that any formula $\varphi$ describing a valid inference from the presence of $x$ is logically implied by $\tilde{\rho}_j$:
\[ (v_K(\tilde{x} \rightarrow \varphi) = \text{TRUE for all } K \in \mathcal{K}) \Rightarrow \tilde{\rho}_j \rightarrow \varphi = \text{TRUE}. \]

In sum, then, in a knowledge space $\mathcal{K}$ on $X$ there are for any $x \in X$ two formulae, $\tilde{\sigma}_j$ and $\tilde{\rho}_j$, describing the inferences from $x$, and each must be implied by the other: in (22) take $\psi = \tilde{\rho}_j$ and in (26) $\varphi = \tilde{\sigma}_j$ to show that $\tilde{\sigma}_j = \tilde{\rho}_j$. A closer look at (21) and (20) shows that all formulae $\tilde{\sigma}_j$ are in disjunctive normal form, while, by (25) and (24), all $\tilde{\rho}_j$ are in conjunctive normal form. One may conclude that surmise mappings and entail relations are in a sense very similar: both collect for any $x \in X$ the inferences that can be made from the presence of $x$ in a state of the corresponding space. Choosing the disjunctive
normal form for these inferences leads to a representation of the space by a surmise mapping, while choosing the conjunctive normal form leads to an entail relation. This also shows that both these representations are alternative formalizations of the “AND/OR graph” in literature on Artificial Intelligence (e.g. Nilsson, 1971; see also Doignon and Falmagne, 1985). It is easy to switch from one form to the other by using the distributive laws of logic, thus making the transition between the two representations on a syntactical level.

This direct correspondence may be illustrated on a small example concerning the following knowledge space $\mathcal{K}$ on $X = \{a, b, c, d\}$:

$$K = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}.$$

The corresponding surmise mapping $\sigma$ can easily be found by checking, for the various $x \in X$, the minimal states containing $x$. As for the entail relation $\mathcal{P}$, the collections $\rho(x)$ of minimal subsets $A$ such that $A \mathcal{P} x$—which are sufficient in (23)—can also be found from inspection of $\mathcal{K}$ by applying (15). The relevant data are collected in the following table, where also the corresponding formulae $\tilde{\sigma}_x$ and $\tilde{\rho}_x$ have been computed:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sigma(x)$</th>
<th>$\tilde{\sigma}_x$</th>
<th>$\tilde{\rho}_x$</th>
<th>$\rho(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>${a}$</td>
<td>$\tilde{a}$</td>
<td>$\tilde{a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${b}$</td>
<td>$\tilde{b}$</td>
<td>$\tilde{b}$</td>
<td>${b}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${a, c}, {b, c}$</td>
<td>$(\tilde{a} \land \tilde{c}) \lor (\tilde{b} \land \tilde{c})$</td>
<td>$(\tilde{a} \lor \tilde{b}) \land \tilde{c}$</td>
<td>${a, b}, {c}$</td>
</tr>
<tr>
<td>$d$</td>
<td>${a, b, d}, {b, c, d}$</td>
<td>$(\tilde{a} \land \tilde{b} \land \tilde{d}) \lor (\tilde{b} \land \tilde{c} \land \tilde{d})$</td>
<td>$(\tilde{a} \lor \tilde{c}) \land \tilde{b} \land \tilde{d}$</td>
<td>${a, c}, {b}, {d}$</td>
</tr>
</tbody>
</table>

Indeed, by the distributive laws, all the $\tilde{\rho}_x$ can be computed directly from $\tilde{\sigma}_x$ and vice versa, moving between the surmise mapping and the entail relation without any appeal to an implied knowledge space.

8. A survey of the various representations

To conclude, an overview is given of the 1–1 correspondences that have been established between (mathematical) objects of different kinds. The basic model includes the conceptualization of a domain of knowledge as a finite set $X$, the knowledge state of a student as the subset of problems this student is capable of solving, and the knowledge structure as the collection of all possible such knowledge states. Thus, a knowledge structure on $X$ is a subset of the power set of $X$.

In order to find alternative characterizations some restriction has to be imposed, so the knowledge space was defined as a structure closed under union, i.e., satisfying Axiom (1). Section 2 presented the 1–1 correspondence between such knowledge spaces and surmise mappings, i.e., the mappings from $X$ into the power set of the power set of $X$ satisfying Axioms (6). In Section 6 another 1–1 correspondence was recalled, that between knowledge spaces and entail relations, which are the class of binary relations on the power set of $X$ satisfying Axioms (14). These three concepts—knowledge spaces, surmise mappings and entail relations—are represented in Fig. 1 by the large boxes, and
equivaleces between them by the two-sided arrows connecting these boxes. An explanation of the horizontal arrow, which is implied by the other two, was provided in Section 7.

In the domain of knowledge spaces interesting special cases were considered. To avoid trivial formal complications, it is assumed throughout that $X$ is the set of notions of the field under investigation, that is, there are no indistinguishable elements in $X$.

First, a knowledge space may be well-graded, which according to Lemma 4.4 is obtained by adding Axiom (13) to (1). Alternatively, a knowledge space may be closed under intersection, which means adding to (1) Axiom (2). It turns out that, under (1), (13) is implied by (2): any (discriminating) knowledge space that is closed under intersection is well-graded (Falmagne and Doignon, 1988b, p. 238). Accordingly, the medium sized box in the top box of Fig. 1 denotes the subcollection of well-graded knowledge spaces and the smallest box the family of knowledge spaces closed under intersection.

Now consider in the domain of surmise mappings the extra conditions corresponding to the subclasses of knowledge spaces defined by (13) and (2), respectively. In Section 4
it was established that the well-graded knowledge spaces are in 1–1 correspondence with the exclusive surmise mappings, obtained by supplementing the Axioms (6) with Axiom (9). The surmise mappings corresponding to knowledge spaces closed under intersection were known from the outset; these are the mappings that essentially coincide with a partial order (quasi order if one wants to allow for indistinguishable elements), that is, the surmise mappings satisfying Axiom (12). The lower left box in Fig. 1 contains boxes representing the collection of surmise mappings defined by (9) and (12), with the appropriate arrows to the top boxes.

For the entail relations, as yet no easy, direct axiom defining the subclass that corresponds to well-graded knowledge spaces and exclusive surmise mappings is known. Conditions deriving from the correspondence with knowledge spaces or surmise mappings can be formulated, but these are indirect and as such not very satisfying. For instance, by Theorem 6.1, $X \setminus A$ is a state in the space $\mathcal{X}$ if and only if in the corresponding entail relation $\mathcal{P}$ we have $A \mathcal{P} Y$ only for $Y \subseteq A$. Via this substitution (13) translates into

\[
(A \neq X \land (A \mathcal{P} Y \Rightarrow A \supseteq Y) \text{ for all } Y \subseteq X) \Rightarrow
\]

for some $x \in X \setminus A$, all $Y \subseteq X$: $(A \cup \{x\}) \mathcal{P} Y \Rightarrow (Y \subseteq A \text{ or } Y = A \cup \{x\})$. (27)

This is indeed rather cumbersome and too close to (13) to be enlightening. The open problem of characterizing the entail relations corresponding to the well-graded knowledge spaces has some practical significance. A solution of this problem would open up the possibility of extending the procedure of Koppen (1993), that constructs knowledge spaces for domains by querying experts about the entail relations, in such a way that only well-graded spaces are produced. This would be desirable, since the requirement of well-gradedness is such a convincing one for knowledge spaces in practice.

The subclass of entail relations corresponding to the knowledge spaces closed under intersection, on the other hand, poses no problems. As with surmise mappings, these are the entail relations that are essentially partial orders on $X$ (quasi orders for indistinguishable elements) and are obtained by adding to (14) the Axiom (18). In Fig. 1 the appropriate boxes (in the lower right box) and the connecting arrows are again drawn. This completes the description of Fig. 1; it presents a concise picture of the various theoretical concepts that played a role in the preceding sections, and their interrelationships.

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