Illumination-invariance of Plateau’s midgray

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Abstract

In a classical experiment Plateau [(1872). Sur la mesure des sensations physiques, et sur la loi qui lie l’intensité de la cause excitante. Bulletins de l’Academie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique, 2me Série, 33, 376–388] found that the midgray operation is illumination-invariant. Falmagne [(1985). Elements of psychophysical theory. New York: Oxford University Press] provided a formalization of this finding that gives rise to two functional equations, the solutions of which determine the possible forms of the psychophysical function. This approach is generalized so that it explicitly refers to the illuminance, and can be applied to the experimentally well-studied class of center-surround stimuli. It is shown that the illumination-invariance of both cross-context matches and Plateau’s midgray specify the psychophysical function up to an increasing function of the relative luminance of stimulus/illumination and center/surround, respectively. Relating this result to that implied by Wallach’s ratio principle clarifies the role of Plateau’s midgray in experimental research on achromatic color perception. Further generalizing the framework extends its scope to more complex experimental settings and the therein observed invariances.

Keywords: Bisection; Brightness; Lightness; Achromatic color perception; Psychophysical invariances; Functional equations

1. Introduction

In 1872 M.J. Plateau reported the following experiment. He provided each of eight artists with a white and a black disk and instructed them to return to their respective studios and paint a gray disk in appearance midway between the other two. Although the illumination conditions will have differed substantially across the studios, the resulting disks were virtually identical (Plateau, 1872; Laming & Laming, 1996).

Generalizing this observation to arbitrary pairs of black, gray, or white disks, and arbitrary illuminances, Falmagne (1985) set up a system of two functional equations. These equations restrict the possible forms of the psychophysical function for the underlying perceptual attribute. In this formalization each of the disks under consideration is identified with the amount of light reflected by it under a fixed standard illuminance. With the symbols $a, b, \ldots$ representing these intensities the disk appearing midway between $a$ and $b$ is denoted by $M(a,b)$. Illumination changes are represented by multiplications with a positive constant $\lambda$, such that the amount of light reflected by a given disk $a$ changes to $\lambda a$. With the above conventions, Falmagne’s first equation

$$\lambda M(a,b) = M(\lambda a, \lambda b).$$

is intended to be a mere restatement of Plateau’s finding, and characterizes the illumination-invariance of the midgray operation. It asserts that forming the midgray after changing the illuminance, which corresponds to the right-hand side of the equation, leads to the same result as first forming the midgray, and changing the illuminance afterwards. Formally, Eq. (1) states that $M$ is a homogeneous function (e.g. Falmagne, 1985). The second equation is nothing but a straightforward formalization of the instruction given (see Fig. 1): The perceived intensity of the midgray $M(a,b)$, specified by the value of a psychophysical function $u$, has to be half-way between those of the stimuli $a$ and $b$, i.e.

$$u[M(a,b)] = \frac{u(a) + u(b)}{2}.$$

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As shown by Falmagne (1985; Theorem 3.12), the only strictly increasing, continuous solutions of Eqs. (1) and (2), given the domain of \( u \) forms an open interval \( I \) of positive real numbers and \( 1 \in I \), are the logarithmic function
\[
u(a) = x \log a + \beta
\]
and the power function
\[
u(a) = xa^\beta + \gamma.
\]
This result is a striking demonstration of how the theory of functional equations can be used in order to derive the form of a particular psychophysical function from very basic empirical observations. Research on achromatic color perception apparently does not make use of Falmagne’s approach, however. Plateau’s midgray operation is not widely used as an experimental task, although it is essential for deriving the above result. Experimental as well as theoretical work nearly exclusively relies on cross-context matches (i.e. setting the intensity of a patch in a test stimulus so that it is perceptually indifferent to a patch in a given standard stimulus). This lack of reception is possibly due the fact that Falmagne’s formalization is based on assumptions which may be valid in complex natural scenes, but probably not in somewhat reduced experimental situations. In general, the appearance of color stimuli depends on their own physical properties as well as on those of the spatial context and the illuminance. Characterizing a stimulus by a single number, as in the above formalization, tacitly assumes that the respective percept is largely independent of those situational aspects. This may be adequate for the viewing conditions as in Plateau’s original study, but not for most of the experimental displays. In the present paper we thus generalize Falmagne’s approach. Various interesting cases can be covered by representing a stimulus configuration by a pair of intensities. These include a homogeneous surface presented under a certain illuminance (independent of the spatial context), and the important class of simple center-surround stimuli consisting of a patch in a surround. In the sequel we will mainly stick to the latter interpretation. In any case, within this general framework we are able to study the implications that illumination-invariance imposes on the possible form of the psychophysical function.

The use of center-surround stimulus configurations for studying achromatic color perception dates back at least to Hess and Pretori (1894/1970). They constitute the minimal relational stimuli that allow for evoking color appearances which cannot be produced by single homogeneous patches (Evans, 1964; Mausfeld & Niedereé, 1993). Fig. 2a shows a display of two center-surround stimuli. Such a side-by-side display is typically presented in an otherwise dark context, and the subject is required to adjust the intensity of one of the patches to match that of the other patch (in some sense specified below).

A different experimental setup is illustrated in Fig. 2b. The stimulus configurations form monocular half-images which are presented in such a way that the surrounds are haploscopically superimposed (e.g. Whittle & Challands, 1969). The percept then consists of two patches in a homogeneous surround. It does not come as a surprise that the data obtained with these two modes of presentation differ substantially. Consider the special case of stimulus configurations in which patches and surrounds are of identical intensity. In a side-by-side display the two patches definitely look different, while in the haploscopic presentation with superimposed surrounds the percept is a homogeneous surface, and so the two patches look the same. This particular limiting case will be central to the theoretical development exposed below.
Available data clearly show that the results not only depend on the mode of presentation, but also on the type of stimuli. The visual system apparently distinguishes between stimulus configurations in which the patch is more intense than the surround (increment) from those with the patch being less intense than the surround (decrements). Moreover, Arend and Spehar (1993a,b) show that the type and degree of invariance exhibited by the data heavily depends on the perceptual criterion. Among the most widely used criteria focusing on certain aspects of the complex percept are lightness (apparent reflectance), brightness (apparent luminance), and brightness contrast (brightness differences). Due to the inherent ambiguity of center-surround configurations not all of these criteria are independent with this kind of stimuli (cf. Arend & Spehar, 1993b). These problems can be resolved by using more complex stimulus configurations (Arend & Spehar, 1993a). In any case, the subsequent results apply whenever the invariance properties they rest upon are empirically valid.

2. Preliminaries

2.1. Stimuli

The set of stimuli $\mathcal{S} \subseteq I^2$ consists of pairs $(a, s)$, each formed by a patch $a$ in a surround $s$, where $I$ is an open interval $(\xi, \xi')$ with $0 < \xi < 1 < \xi'$ (see Fig. 2). The pair $(a, s)$ then represents the stimulus $(a, s)$ after changing illuminance by a factor $\lambda > 0$.

As already mentioned above, consideration of the following three cases is motivated by empirical results indicating that the visual system distinguishes between incremental and decremental patches. First, the set of stimuli is assumed to be the full Cartesian product $I^2$. Second, we consider the subset of incremental stimuli, or increments formed by the set $\mathcal{S} = \{(a, s) \mid a, s \in I, a \geq s\}$. The cases $(s, s)$, where the intensity of the patch equals that of the surround, are included for technical reasons. They play an essential role in the subsequent derivations. Third, the subset of decremental stimuli, or decrements is defined to be the set $\mathcal{D} = \{(a, s) \mid a, s \in I, a \leq s\}$, where we again include equality. Moreover, set $\mathcal{D}$ describes the case where the pair $(a, s)$ specifies the luminance $a$ of a homogenous surface under illuminance $s$. In the sequel $\mathcal{S}$ can be identified with either $I^2$, $\mathcal{S}$, or $\mathcal{D}$.

2.2. Empirical structure and representation

Traditionally, effects of illuminance or surround on the appearance of color stimuli are studied by means of cross-context comparisons. Thus, we consider a binary relation $\succ$ on $\mathcal{S}$, for which several interpretations are possible, as already indicated. In general, $(a, s) \succ (b, t)$ means that patch $a$ presented within surround $s$ is perceived as being at least as intense (according to one of the above mentioned criteria) than patch $b$ presented within surround $t$. Most of the time only matches $(a, s) \sim (b, t)$ are considered. By $\sim$ we denote the symmetric part of $\succ$, i.e. $(a, s) \sim (b, t)$ if and only if $(a, s) \succ (b, t)$ and $(b, t) \succ (a, s)$. The different possible interpretations of $\succ$ indicate that $\sim$ is not a metameric match. For the subsequently developed arguments only the formal properties of $\succ$ and $\sim$ matter, but not their interpretation.

To reflect the fact that Plateau’s midgray is formed under a fixed illuminance, we introduce a midgray operation $M$, assigning to all $(a, s), (b, s) \in \mathcal{S}$ a patch $M(a, b)$ within surround $s$ which is judged to appear “midway between” the patches $a$ and $b$. Thus, the midgray is formed within a certain surround, and not across contexts.

Let us assume that there exists a psychophysical function $u: \mathcal{S} \to \mathbb{R}$, strictly increasing in the first argument, and continuous in both arguments. Generalizing Falmagne’s representation we require that

$$(a, s) \succ (b, t) \quad \text{iff} \quad u(a, s) \geq u(b, t)$$

(3)

for all $(a, s), (b, t) \in \mathcal{S}$, and

$$u[M(a, b), s] = \frac{u(a, s) + u(b, s)}{2}$$

(4)

for all $(a, s), (b, s) \in \mathcal{S}$.

Notice that only the analogue of Eq. (4) was stated in Falmagne’s case, because the psychological relation was assumed to coincide with the physical ordering of the patches.

Although it is not the primary goal of the paper to investigate the empirical conditions that underly a representation $u: \mathcal{S} \to \mathbb{R}$ satisfying Eqs. (3) and (4), we shortly touch upon this problem. In general, the desired representation can be established in two steps. First, for a given surround $s$ standard results (e.g. Theorem 6.11 of Krantz, Luce, Suppes, & Tversky, 1971) may be used to identify sufficient conditions for a context-specific representation $u_s(a) = u(a, s)$. Among these conditions are bisymmetry, idempotency and commutativity of the midgray operation $M_s$, and its compatibility to the within-context ordering induced by $\succ$. Second, a global representation $u$ is obtained by exploiting the non-uniqueness of the interval-scale representations $u_s$ in order to make them coincide on cross-context matches.

The monotonicity assumption on $u$ requires that we have

$$(a, s) \succ (b, s) \quad \text{iff} \quad a \geq b$$

(M1)

for all $(a, s), (b, s) \in \mathcal{S}$. The subsequent derivations will heavily draw upon (M1), which allows that for fixed $s \in I$ we can invert the function $u_s(a) = u(a, s)$. Assuming that the psychophysical function $u$ is strictly decreasing in the second argument amounts to postulating

$$(a, s) \succ (a, t) \quad \text{iff} \quad s \leq t$$

(M2)

for all $(a, s), (a, t) \in \mathcal{S}$. Property (M2) states that the perceived intensity of a patch is inversely monotonic with the physical intensity of the surround, a phenomenon which is known as simultaneous brightness contrast. The
subsequently presented results, however, follow without assuming (M2).

3. Illumination-invariance

As in Falmagne (1985) changes of illumination are represented by multiplying intensities with a positive constant. Conditions (I1) and (I2) below assert that cross-context matches and the midgray operation are invariant under these illumination changes.

Cross-context matches \( \sim \) on \( \mathcal{I} \) are said to be illumination-invariant if for each real constant \( \lambda > 0 \) we have

\[
(a, s) \sim (b, t) \quad \implies \quad (\lambda a, \lambda s) \sim (\lambda b, \lambda t),
\]

(III1)

whenever \( (a, s), (b, t), (\lambda a, \lambda s), (\lambda b, \lambda t) \in \mathcal{I} \).

Fig. 3 provides an illustration of property (III1) in log–log coordinates. If this condition holds then any two indifference curves can be made to coincide by a translation along the diagonal. Notice that the form of the indifference curves is not uniquely determined by illumination-invariance of cross-context matches. The schematic indifference curves in Fig. 3 resemble those obtained by Whittle and Challands (1969).

The midgray operation \( M_s, s \in I \), is said to be illumination-invariant if for each real constant \( \lambda > 0 \)

\[
\lambda M_s(a, b) = M_s(\lambda a, \lambda b)
\]

(I12)

whenever \( (a, s), (b, s), (\lambda a, \lambda s), (\lambda b, \lambda s) \in \mathcal{I} \). Notice that the equation in (I12) corresponds to the perceptual indifference

\[
(\lambda M_s(a, b), \lambda s) \sim (M_s(\lambda a, \lambda b), \lambda s),
\]

which matches two stimuli presented under the same illumination conditions. This adds a new perspective to research on lightness perception, the standard paradigm of which is matching across different simultaneously available illumination conditions. Section 4 will provide a detailed discussion of this issue.

With the subsequently presented result we are able to follow the line of reasoning in Falmagne’s proof (Falmagne, 1985) in the considered generalized situation, too. Notice that the theorem only refers to an ordinal representation of the cross-context relation.

**Theorem 1.** Let the function \( u \) from \( \mathcal{I} \) into the reals represent the weak order \( \preceq \) on \( \mathcal{I} \) according to (3), and let \( \preceq \) satisfy property (M1). Then the following two statements are equivalent:

1. Cross-context matches are illumination-invariant, i.e. property (I11) holds.
2. For each real constant \( \lambda > 0 \)

\[
u_s[\lambda u_s^{-1}(p)] = u_s[\lambda u_s^{-1}(p)]
\]

whenever there are \( (a, s), (b, t) \in \mathcal{I} \) such that \( u(a, s) = u(b, t) = p \), and \( (\lambda a, \lambda s), (\lambda b, \lambda t) \in \mathcal{I} \).

**Proof.** For proving the implication from 1 to 2, let \( (a, s), (b, t) \in \mathcal{I} \) such that \( u(a, s) = u(b, t) = p \), and \( (\lambda a, \lambda s), (\lambda b, \lambda t) \in \mathcal{I} \). By (3) we get \( (a, s) \sim (b, t) \), and from (I11) we then infer that \( (\lambda a, \lambda s) \sim (\lambda b, \lambda t) \) holds. This is equivalent to \( u(\lambda a, \lambda s) = u(\lambda b, \lambda t) \), or \( u_s(\lambda a) = u_s(\lambda b) \) again by (3). Substituting \( a = u_s^{-1}(p) \) and \( b = u_s^{-1}(p) \) provides the assertion.

Conversely, choose any \( \lambda > 0 \), and assume that \( (a, s), (b, t), (\lambda a, \lambda s), (\lambda b, \lambda t) \in \mathcal{I} \) such that \( (a, s) \sim (b, t) \). Eq. (3) then provides \( u(a, s) = u(b, t) = p \), i.e. \( a = u_s^{-1}(p) \) and \( b = u_s^{-1}(p) \). We thus have \( u_s(\lambda u_s^{-1}(p)) = u_s(\lambda u_s^{-1}(p)) \) by 2 which is equivalent to \( (\lambda a, \lambda s) \sim (\lambda b, \lambda t) \) according to (3).

Now, assume that \( u \) is as specified in Section 2.2, and satisfies Eqs. (3) and (4). In order to see how the result provided by Theorem 1 drives the argument, consider \( (a, s), (b, s), (\lambda a, \lambda s), (\lambda b, \lambda s) \in \mathcal{I} \) such that \( u(a, s) = u(b, s) = p \). Using the notation \( u_s \) already introduced, Eq. (4) implies

\[
M_s(a, b) = u_s^{-1}\left[\frac{u_s(a) + u_s(b)}{2}\right],
\]

which after inserting into (I12) provides

\[
\lambda u_s^{-1}\left[\frac{u_s(a) + u_s(b)}{2}\right] = u_s^{-1}\left[\frac{\lambda u_s(a) + \lambda u_s(b)}{2}\right].
\]

With \( p = u_s(a) \) and \( q = u_s(b) \) we get

\[
u_s\left[\frac{\lambda u_s^{-1}(p) + u_s^{-1}(q)}{2}\right] = u_s\left[\frac{\lambda u_s^{-1}(p)}{2}\right].
\]

(5)

Keeping \( \lambda \) fixed for the moment, by Theorem 1 we can define

\[
v_s(p) = u_s[\lambda u_s^{-1}(p)]
\]

![Fig. 3. Illustration of illumination-invariance of cross-context matches (I11) in log–log coordinates. Solid lines represent indifference curves.](image-url)
with a strictly increasing and continuous function \( v_\lambda \) that is independent of \( s \). To determine the domain of \( v_\lambda \) consider the open interval \( I_\lambda = (\zeta / \lambda, \zeta / \lambda) \). The set \( \mathcal{S}_\lambda = I_\lambda \cap \mathcal{S} \) then consists of all stimuli \((a, s)\) in \( \mathcal{S} \), for which \((\lambda a, \lambda s)\) is in \( \mathcal{S} \), too. Notice that \( \mathcal{S}_\lambda \) is empty whenever \( \zeta \leq \zeta / \lambda \) or \( \zeta \leq \lambda \zeta \). The domain of \( v_\lambda \) then is the image of \( \mathcal{S}_\lambda \) under the function \( u \), which we denote by \( U_\lambda \). The set \( U_\lambda \) is a real interval, because \( \mathcal{S}_\lambda \) is (path-)connected, and \( u \) is assumed to be continuous.

From Eq. (5) it then follows that the function \( v_\lambda: U_\lambda \to \mathbb{R} \) satisfies

\[
\frac{v_\lambda(p) + v_\lambda(q)}{2} = \frac{v_\lambda(p + q)}{2}.
\]

Under the given conditions, the solution of this functional equation, which is known as the Jensen equation, is

\[
v_\lambda(p) = f_\lambda \cdot p + g_\lambda,
\]

(e.g. Aczél, 1987, Corollary 3, p. 127), where \( f_\lambda > 0 \) and \( g_\lambda \) are real constants that depend on the choice of \( \lambda \).

If we substitute \( u \), and let \( \lambda \) vary again then there results

\[
u(a, \lambda s) = f(\lambda) \cdot u(a, s) + g(\lambda),
\]

where \((a, s), (\lambda a, \lambda s) \in \mathcal{S}, \lambda > 0 \), and functions \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) and \( g: \mathbb{R}^+ \to \mathbb{R}^+ \). We subsequently provide the solutions of Eq. (7) under two different conditions that are related to the experimental setups discussed above.

If we consider the special case \( a = s = 1 \) in Eq. (7) then we obtain

\[
u(\lambda, \lambda) = f(\lambda) \cdot u(1, 1) + g(\lambda).
\]

Subtracting this equation from (7), and defining \( w(a, s) = u(a, s) - u(1, 1) \) which is possible because we assume \((1, 1) \in \mathcal{S}\), provides us with

\[
w(\lambda a, \lambda s) = f(\lambda) \cdot w(a, s) + w(\lambda, \lambda).
\]

Setting \( a = s \) finally yields

\[
w(\lambda s, \lambda s) = f(\lambda) \cdot w(s, s) + w(\lambda, \lambda).
\]

Notice that Eqs. (8) and (9) are vacuous if there exists a real constant \( c \) such that \( u(s,s) \equiv c \) for all \( s \in I \). This constancy will hold for presentations with haploscopically superimposed surrounds. Thus, before providing the solutions of Eq. (7) under the assumption \( u(s,s) \equiv c \), we first treat the case \( u(s,s) \) not constant. Considering side-by-side displays we may even assume that \( u(s,s) \) is strictly increasing with \( s \), which seems to be reasonable at least for certain perceptual criteria, like brightness.

### 3.1. The case \( u(s,s) \) not constant

If \( f(\lambda) \equiv 1 \) then, setting \( l(s) = w(s, s) \), by Eq. (9) we obtain

\[
l(\lambda s) = l(\lambda) + l(s).
\]

with \( \lambda, s, \lambda s \in I \). Reducing this equation to the restricted Cauchy equation we may apply standard results (e.g. Aczél, 1987, Section 1, Corollary 8) to prove that there is a real constant \( z \neq 0 \) such that

\[
l(s) = z \log s,
\]

where \( \log \) denotes the natural logarithm. In case of \( l \) strictly increasing we even have \( z > 0 \).

Notice that this only restricts the form of \( w(\lambda, \lambda) \), but not of \( w \) in general. In order to determine the consequences for \( w(a, s) \) we consider

\[
w(\lambda a, \lambda s) = w(a, s) + l(\lambda),
\]

from which we derive

\[
w(a, s) = w\left(s \cdot \frac{1}{a}, s \cdot \frac{1}{s}\right) = w\left(\frac{a}{s}, 1\right) + l(s).
\]

If we define \( h(x) = w(x, 1) \) and resubstitute \( u \) then we obtain

\[
u(a, s) = h\left(\frac{a}{s}\right) + z \log s + \beta
\]

with real constants \( z \neq 0 \) and \( \beta \) (\( z > 0 \) whenever \( u(s,s) \) strictly increasing).

Now, let \( f(\lambda) \equiv 1 \). Then there is \( \lambda_0 > 0 \) such that \( f(\lambda_0) \neq 1 \).

Since Eq. (9) is symmetric with respect to \( \lambda \) and \( s \) we get

\[
 f(\lambda_0) \cdot w(s, s) + w(\lambda_0, \lambda) = f(s) \cdot w(\lambda, \lambda) + w(s, s).
\]

Thus for \( \lambda_0 \) we have

\[
 w(s, s) \cdot [f(\lambda_0) - 1] = w(\lambda_0, \lambda_0) \cdot [f(s) - 1].
\]

Defining

\[
\mu = \frac{w(\lambda_0, \lambda_0)}{f(\lambda_0) - 1}
\]

then provides

\[
w(s, s) = \mu[f(s) - 1].
\]

We have \( \mu = 0 \), because \( u \) and thus \( w \) is not constant. If we plug (10) into (9) we obtain

\[
\mu[f(s) - 1] = \mu f(s) [f(s) - 1] + \mu[f(s) - 1],
\]

which simplifies to

\[
f(\lambda s) = f(s) f(s)
\]

with \( \lambda, s, \lambda s \in I \). We may again reduce this restricted multiplicative Cauchy equation to the additive case, and apply standard results (e.g. Aczél, 1987, Section 1, Corollary 8), to obtain

\[
f(s) = e^x
\]

with a real constant \( x \).

If we set

\[
w''(a, s) = w(a, s) + \mu
\]

and remember that \( w(\lambda, \lambda) = \mu[f(\lambda) - 1] \) then (8) goes over into

\[
w'(\lambda a, \lambda s) = f(\lambda)w'(a, s).
\]
Thus
\[ w^a(a,s) = w^a \left( s \cdot \frac{1}{s} \cdot a, s \cdot \frac{1}{s} \right) = f(s) \cdot w^a \left( \frac{a}{s}, 1 \right). \]

Defining \( h(x) = w^a(x, 1) \) and resubstituting \( u \) then provides
\[ u(a,s) = h \left( \frac{a}{s} \right) \cdot s^2 + \beta \]
with real constants \( \alpha, \beta \) (\( \alpha > 0 \) whenever \( u(s,s) \) strictly increasing).

3.2. The case \( u(s,s) \) constant

Setting \( a = s \), and assuming \( u(s,s) \equiv c \), we get
\[ c = c \cdot f(\lambda) + g(\lambda) \]
from (7). If we subtract this equation from (7) then, with \( w(a,s) = u(a,s) - u(1,1) = u(a,s) - c \) as above, we obtain
\[ w(\lambda a, \lambda s) = f(\lambda) \cdot w(a,s), \quad \text{(13)} \]
where \( (a,s), (\lambda a, \lambda s) \in \mathcal{S} \) and \( \lambda > 0 \). Notice that \( f \neq 0 \), because \( u \) (and thus \( w \)) is assumed to be non-constant. From (13) we get
\[ w \left( s \cdot \frac{a}{s}, s \cdot \frac{1}{s} \right) = w(a,s) = f(s) \cdot w \left( \frac{a}{s}, 1 \right). \]

This equation together with
\[ w(\lambda a, \lambda s) = f(\lambda s) \cdot w \left( \frac{a}{s}, 1 \right) \]
and (13) provides the restricted multiplicative Cauchy (11). Proceeding as above we get the solution (12). Specifying constants by setting \( a = s \), however, yields \( \lambda = 0 \), because we assume \( u(s,s) \equiv c \). Thus, we have
\[ u(a,s) = h \left( \frac{a}{s} \right) + \beta \]
with a real constant \( \beta \).

3.3. Summary of results

The following theorem summarizes the results obtained in the previous sections. Notice that additive constants do not have any empirical meaning, because \( u \) is an interval-scale representation.

**Theorem 2.** Let the function \( u: \mathcal{S} \to \mathbb{R} \) be strictly increasing in the first argument, and continuous in both arguments. Assume that it satisfies Eqs. (3) and (4), and the illumination invariance properties (II1) and (II2).

If \( u(s,s), s \in I \), is not constant then \( u \) is of the form
\[ u(a,s) = h \left( \frac{a}{s} \right) + \alpha \log s + \beta \quad \text{(14)} \]
with a strictly increasing and continuous function \( h \) and real constants \( \alpha \neq 0 \) and \( \beta \) (\( \alpha > 0 \) whenever \( u(s,s) \) strictly increasing), or
\[ u(a,s) = h \left( \frac{a}{s} \right) \cdot s^2 + \beta \quad \text{(15)} \]
with a strictly increasing and continuous function \( h \) and a real constant \( \beta \).

4. Discussion

The present paper generalizes Falmagne’s (1985) formalization of the illumination-invariance observed in Plateau’s classical experiment (Plateau, 1872; Laming & Laming, 1996). Instead of representing stimuli by a single number, the suggested framework refers to pairs \((a, s)\) that may be interpreted in different ways. They may characterize the luminance \( a \) of a homogeneous surface presented under illumination \( s \), or center-surround stimuli (patch \( a \) in surround \( s \)), which received considerable attention in research on achromatic color perception. Falmagne’s functional equations are restated in this generalized context (see Eqs. (3) and (4)). Theorem 1 shows that we can adopt the line of reasoning in Falmagne (1985) to solve these equations. Theorem 2 then presents the possible forms of the psychophysical function \( u \). In contrast to Falmagne’s solutions of the functional equations (1) and (2) the form of \( u \) is only determined up to a strictly increasing and continuous, but otherwise unspecified, function \( h \) evaluating the ratio \( a/s \). This normalization amounts to considering the relative luminance in a center-surround stimulus, or the reflectance of a homogeneous surface presented under a certain illumination. In case of \( u(s,s) \) being non-constant (as in side-by-side displays of center-surround stimuli) the value \( h(a/s) \) is either shifted to a level determined by a logarithmic function of \( s \), or multiplied by a power function of \( s \) (Eqs. (14) and (15)). Whereas in case of \( u(s,s) \) being constant there is a complete disconnecting of the context (illumination, or surround, respectively). The psychophysical function (16) implies an even stronger notion of illumination-invariance. Property (II3) states that changing illumination not only preserves cross-context matches, but leaves invariant the perceptual aspect represented by the psychophysical function \( u \). Cross-context matches \( \sim \) on \( \mathcal{S} \) are said to satisfy the ratio principle if for each real constant \( \lambda > 0 \)
\[ (a,s) \sim (\lambda a, \lambda s) \quad \text{II3} \]
holds whenever \((a,s), (\lambda a, \lambda s) \in \mathcal{S}\). Wallach (1948) was the first to formulate the ratio principle for center-surround stimuli. Notice that this property completely specifies the form of the indifference curves, which in log–log coordinates are straight lines with unit slope (see Fig. 4).

Although the solutions listed in Theorem 2 induce different shapes of the cross-context indifference curves,
for the midgray operation they all imply

\[ M_1(a,b) = s \cdot h^{-1}\left[ \frac{h(u)}{2} \right] \]

with a strictly increasing and continuous function \( h \).

Discussing the obtained results in the light of available data provides a new perspective on Falmagne’s approach, and on its lack of reception in experimental research. Jacobsen and Gilchrist (1988a,b) found the ratio principle for center-surround stimuli to hold over a large range of intensities. They considered contradicting empirical evidence (e.g. Hess & Pretori, 1894/1970; Jameson & Hurvich, 1961) to be due to experimental artifacts. Given the ratio principle (I13) we may conclude that, even within a purely ordinal approach based on Eq. (3), the psychophysical function \( u \) is of the form \( u(a,s) = h(a/s) \) with some strictly increasing function \( h \). Noticing that the ratio principle implies \( u(s,s) \) being constant, Theorem 2 seems to suggest that in this case there is no benefit from additionally considering Plateau’s midgray operation. Moreover, the illumination-invariance (I12) of Plateau’s midgray formally follows from the ratio principle (I13) through Eqs. (3), (4) and monotonicity (M1). At first sight, this may be considered as providing a theoretical post-hoc justification for basing research exclusively on cross-context matching.

In fact, however, conditions (I13) and (I12) express different types of illumination-invariance. Cross-context matching in (I13) refers to a situation where the viewer directly compares the percepts arising from simultaneously available illumination conditions, as in case of the inhomogeneous illumination occurring across a shadow boundary, for example. On the contrary, the midgray operation in (I12) provides a means for relating illumination conditions that are not presented simultaneously, but successively. This includes situations where the illumination changes in time, as is the case when perceiving the same objects under varying daylight conditions. This is exactly what Plateau’s original experiment was about. Adapting the terminology introduced by Brainard, Brunt, and Speigle (1997) in the context of color constancy, we use the term simultaneous illumination-invariance to refer to invariance under illumination changes occurring across space and the term successive illumination-invariance to refer to invariance under illumination changes occurring in time. Previous research on the perception of achromatic colors nearly exclusively focused on simultaneous illumination-invariance. The presented approach thus adds a new perspective to the research on lightness perception. If applied to center-surround configurations, conditions (I11) and (I12) formalize successive illumination-invariance of cross-context matches and the midgray operation, respectively. In this context the midgray operation can take over the role that experimental tasks such as determining achromatic loci (Brainard, 1998) play in research on color constancy.

It cannot be expected that one type of illumination invariance implies the other. They may even be mediated by distinct visual mechanisms (Brainard et al., 1997). This point of view is partially corroborated by Heller (2001), who conducted an experimental test of the illumination-invariance of the midgray operation for decremental center-surround configurations. Instead of directly producing the perceived midgrays, they were determined indirectly by applying the method of constant stimuli, which lead to very small standard errors. Statistically testing individual data, condition (I12) turned out to hold for five out of twelve subjects. The clear-cut individual differences seem to rule out an explanation of the resulting violations of (I12) as an artifact due to insufficient control of the observer’s state of adaptation.

Although the theoretical framework introduced above is developed for simple center-surround stimuli it may be generalized to more complex experimental settings. In particular, it is easily extended to apply to stimulus configurations that consist of a fixed spatial layout of homogeneous surfaces. Arend and Spehar (1993a,b), for example, enriched the viewing conditions by embedding center-surround configurations into Mondrian-like patterns. Each stimulus in this kind of setting may be represented by a tuple \((a,s)\), where \( a \) denotes the luminance of the patch that is to be judged, and \( s \) specifies the array of luminances that constitute the context components (e.g. the surround and the elements of the Mondrian-like pattern). It is straightforward to recast Eqs. (3), (4) as well the invariance conditions (I11), (I12), (I13) and monotonicity (M1) within this generalized framework, and a result analogous to that of Theorem 2 can be derived from these properties. Normalization then is with respect to an arbitrary component of s, and is applied to all remaining
components of \((a, s)\). These quotients form the arguments of the extended function \(h\). It is an open question whether illumination-invariance of the midgray operation holds within this type of experimental setting. Moreover, it not clear whether the produced midgrays are susceptible to the instruction provided to the observers in the same way as cross-context matches are. Arend and Spehar (1993a,b) showed that the results in the latter task heavily depend on the used perceptual criterion (e.g. lightness, brightness, or brightness-contrast).

The extended framework can also be modified to capture an invariance property reported by Gilchrist and Jacobsen (1983). In their study observers were asked to match the lightness of identified surfaces from two 3D scenes (an outdoor scene and a scene composed of artificial objects). Viewing these scenes through a veiling luminance, which added a constant amount to all the luminances, was found to have no significant effect on the matchings compared to observation without the veil. This suggests an additive type of invariance instead of the multiplicative invariance captured by (II3). In this case, the implied normalization with respect to one of the context components is done by forming differences rather than quotients. Besides that, with additive versions of (II1) and (II2), the effect of the context is either an additive shift by a linear function of the context component, or a multiplication by an exponential function of it. Deriving these solutions requires the stimulus with zero intensity in all components, rather than that with all intensities equal to 1, to be included in the considered set \(\mathcal{F}\) of stimuli.

The discussion has shown that extending Falmagne’s formalization of Plateau’s experiment (Falmagne, 1985) to the well-studied class of center-surround stimuli not only restricts the possible forms of the psychophysical function in this case, but also has various implications on current research in achromatic color perception. It forms the basis for introducing a distinction between simultaneous and successive illumination-invariance that parallels a classification suggested by Brainard et al. (1997) in the context of color constancy. Plateau’s midgray operation is identified as a means for investigating successive illumination-invariance, which has been neglected in previous research. Moreover, the framework can be extended further to cope with the increased complexity of (at least some of the) more recently used experimental settings. Finally, the approach is able to capture invariance properties other than those governed by the ratio principle, and allows for deriving the constraints they put on the psychophysical function.

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